Proper actions, fixed-point algebras, and deformation via coactions

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Proper actions

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Then: Define a $C_c(G, C_0(X)) \subseteq C_0(X) \rtimes G$-inner product on $C_c(X)$ by

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We get

$$C_0(G \backslash X) \cong \mathcal{K}(\mathcal{F}(X)) \quad \text{via} \quad (f \cdot \xi)(x) = f(Gx)\xi(x).$$

Therefore $\mathcal{F}(X)$ becomes a

$$C_0(G \backslash X) - \overline{\langle \mathcal{F}(X), \mathcal{F}(X) \rangle} \subseteq C_0(X) \rtimes G$$

equivalence bimodule.

Rieffel proper actions

Now let $\alpha : G \curvearrowright A$. Let $L^1_\Delta(G, A) = \Delta^{1/2} L^1(G, A) \cap L^1(G, A)$.
α is Rieffel proper iff there exists a dense subspace $\mathcal{R} \subseteq A$ s.t.

$$\langle \xi, \eta \rangle_{A \rtimes_r G} := [g \mapsto \Delta(g)^{-1/2} \xi^* \alpha_g(\eta)] \in L^1_\Delta(G, A) \quad \forall \xi, \eta \in \mathcal{R}.$$

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$\alpha : G \curvearrowright A$ is called saturated, if $\langle \cdot, \cdot \rangle_{A \rtimes_r G}$ is full!

Some questions about generalized fixed-point algebras

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2. (Rieffel) Is there an analogous theory for maximal crossed products, i.e., an equivalence

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   Answer: Not clear in general, but for a large class of examples, such theory exists!
3. What are the examples?

Weak $X \rtimes G$-algebras

Definition(Buss-E '13) Suppose $G \curvearrowright X$ is proper. A G-C^*-algebra (A, G, α) is called weak $X \rtimes G$-algebra if $\exists$ a G-equivariant, nondegenerate (i.e., $\phi(C_0(X))A = A$) $*$-homomorphism $\phi : C_0(X) \to M(A)$.

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Let $\mathcal{F}_c(A) := \phi(C_c(X))A$. For $\xi = \phi(f)a, \eta = \phi(h)b \in \mathcal{F}_c(A)$ we get

$$\begin{aligned}\langle \xi, \eta \rangle(g) &= \Delta(g)^{-1/2}(\phi(f) \cdot a)^* \alpha_g(\phi(h) \cdot b) = a^* \big(\Delta(g)^{-1/2} \phi(\bar{f} \tau_g(h))\big) b \\ &= a^* \left[\phi \rtimes_\mu G\big(\langle f, h \rangle_{C_0(X) \rtimes G}\big)\right] b\end{aligned}$$

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In particular, $\langle \xi, \xi \rangle = a^* \left[\phi \rtimes_\mu G\left(\langle f, f \rangle_{C_0(X) \rtimes G}\right)\right] a \geq 0$ in $M(A \rtimes_\mu G)$!

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Thus $(\mathcal{F}_c(A), \langle \cdot, \cdot \rangle)$ completes to a $A \rtimes_\mu G$-Hilbert module $(\mathcal{F}_\mu(A), \langle \cdot, \cdot \rangle_{A \rtimes_\mu G})$ for every crossed-product norm $\| \cdot \|_\mu$ on $C_c(G, A)$! If $G \curvearrowright X$ is free, then $\langle \cdot, \cdot \rangle_{A \rtimes_\mu G}$ is full.

The fixed-point algebras

Let $\alpha : G \curvearrowright A$ be a weak $X \rtimes G$-algebra, $A_c := C_c(X) A C_c(X)$.
We define the fixed-point algebra with compact supports as

$$A_c^G := C_c(G\backslash X) \cdot \{m \in M(A)^G : f \cdot m, m \cdot f \in A_c\} \cdot C_c(G\backslash X) \subseteq M(A)^G.$$

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Then A_c^G acts on the left of $\mathcal{F}_c(A)$ via $m \cdot \xi = m\xi \in M(A)$ and there is a compatible A_c^G-valued inner product

$$_{A_c^G}\langle \xi, \eta\rangle = \int_G^{str} \alpha_g(\xi\eta^*)\, dg \quad \text{s.t.} \quad {}_{A_c^G}\langle \xi, \eta\rangle\zeta = \xi\langle \eta, \zeta\rangle_{C_c(G,A)}$$

Therefore the left action induces a $*$-homomorphism $\Psi : A_c^G \to \mathcal{K}(\mathcal{F}_\mu(A))$ with dense image.

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Definition (Buss-E) Let $\| \cdot \|_\mu$ be a crossed-product norm on $C_c(G, A)$. Then we call $A_\mu^G := \overline{\Psi(A_c^G)} = \mathcal{K}(\mathcal{F}_\mu(A))$ the μ-generalized fixed-point algebra for the weak $X \rtimes G$-algebra (A, α). We get $A_\mu^G \sim_M \overline{\langle \mathcal{F}_\mu(A), \mathcal{F}_\mu(A) \rangle} \subseteq A \rtimes_\mu G$.

The case $X = G$ and Landstad duality for coactions

In what follows we equip $C^*(G)$ with the comultiplication $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G)); g \mapsto g \otimes g$.

Recall A coaction of G on B is a $*$-hom. $\delta : B \to M(B \otimes C^*(G))$ such that

$$(\mathrm{id}_B \otimes \delta_G) \circ \delta = (\delta \otimes \mathrm{id}_G) \circ \delta \quad \text{and} \quad \delta(B)(1 \otimes C^*(G)) = B \otimes C^*(G).$$

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Coaction crossed product If (B, δ) is a coaction of G, then let

$$j_B : B \to M(B \otimes \mathcal{K}(L^2(G))); \quad j_B(b) = (\mathrm{id}_B \otimes \lambda) \circ \delta(b)$$
$$j_G : C_0(G) \to M(B \otimes \mathcal{K}(L^2(G))); \quad j_{C_0(G)}(f) = 1 \otimes M_f$$

Then $B \rtimes_\delta \widehat{G} := \overline{j_B(B) j_G(C_0(G))} \subseteq M(B \otimes \mathcal{K}(L^2(G)))$.

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Dual action: $\widehat{\delta} : G \curvearrowright B \rtimes_\delta \widehat{G}$; $\quad \widehat{\delta}_g \big(j_B(b) j_G(f)\big) = j_B(b) j_G(\sigma_g(f))$ with $\sigma : G \curvearrowright C_0(G) : \big(\sigma_g(f)\big)(h) = f(hg)$.

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Thus: $(B \rtimes_\delta \widehat{G}, \widehat{\delta}, \phi = j_{C_0(G)})$ is a weak $G \rtimes G$-algebra!

Dual coactions and duality theorems

Let $\alpha : G \curvearrowright A$ be an action and let $\rtimes_\mu = \rtimes_r$ or $\rtimes_{\max}$ (or any 'suitable' intermediate crossed product). The dual coaction

$$\widehat{\alpha}_\mu : A \rtimes_\mu G \to M(A \rtimes_\mu G \otimes C^*(G))$$

is given by $\quad A \ni a \mapsto i_A(a) \otimes 1 \quad$ and $\quad G \ni g \mapsto i_G(g) \otimes g$
(with $(i_A, i_G) : (A, G) \to M(A \rtimes_\mu G)$ the canonical maps).

Takesaki-Takai duality: $A \rtimes_\mu G \rtimes_{\widehat{\alpha}_\mu} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$.

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Katayama-duality: If (B, δ) is a coaction, we always have a canonical surjection

$$\Psi = (j_B \rtimes j_G) \rtimes (1 \otimes \rho) : B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta},\max} G \to B \otimes \mathcal{K}(L^2(G))$$

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Let $\alpha : G \curvearrowright A$ be an action and let $\rtimes_\mu = \rtimes_r$ or $\rtimes_{\max}$ (or any 'suitable' intermediate crossed product). The dual coaction

$$\widehat{\alpha}_\mu : A \rtimes_\mu G \to M(A \rtimes_\mu G \otimes C^*(G))$$

is given by $A \ni a \mapsto i_A(a) \otimes 1$ and $G \ni g \mapsto i_G(g) \otimes g$ (with $(i_A, i_G) : (A, G) \to M(A \rtimes_\mu G)$ the canonical maps).

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Fixed-point algebras and Landstad duality

Every weak $G \rtimes G$-alg. (A, α, ϕ) is of the form $(B \rtimes_\delta \widehat{G}, \widehat{\delta}, j_{C_0(G)})$!

Theorem (Buss, E. (2014)): Let (A, α, ϕ) be as above. For every intermediate duality crossed-product $A \rtimes_\mu G$ there exists a (unique up to isom.) μ-coaction (B_μ, δ_μ) such that

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Idea of proof Let $\mathcal{F}_\mu(A) = \overline{C_c(G)A}^\mu$ be the $\mathcal{K}(\mathcal{F}_\mu(A)) - A \rtimes_\mu G$ Morita equivalence. Construct a $\widehat{\alpha_\mu}$-compatible coaction $\delta_{\mathcal{F}_\mu}$ on $\mathcal{F}_\mu(A)$ wich then induces a coaction $\delta_{A_\mu^G}$ on $A_\mu^G \cong \mathcal{K}(\mathcal{F}_\mu(A))$. One then checks that $(B_\mu, \delta_\mu) := (A_\mu^G, \delta_{A_\mu^G})$ does the job. □

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Corollary If (B, δ) is a coaction and $(B \rtimes_\delta \widehat{G}, \widehat{\delta}, j_{C_0(G)})$ is the (dual) weak $G \rtimes G$-algebra, then there is a unique intermediate crossed product $(B \rtimes_\delta \widehat{G}) \rtimes_\mu G$ such that $(B, \delta) \cong (B_\mu, \delta_\mu)$.

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Indeed: $\exists$ a unique C^*-norm $\|\cdot\|_\mu$ on $C_c(G, B \rtimes_\delta \widehat{G})$ such that

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Deformation a la Kasprzak, Bhowmick-Neshveyev-Sangha

Rieffel '93 Let $\beta : \mathbb{R}^n \curvearrowright B$ be an action and let $J \in M_n(\mathbb{R})$ skew symmetric. Using β, Rieffel constructs a new (deformed) multiplication $*_J$ on some subalgebra $\mathcal{S}(B) \subseteq B$ and obtains a deformed C^*-algebra $B_J := \overline{\mathcal{S}(B)}$.

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He constr. a deformed system (B^ω, β^ω) s.t.

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Yamashita '11 (G discrete), Bhowmick-Neshveyev-Sangha '13
Extend this deformation procedure to possibly non-abelian groups by replacing actions (B, β) of $\widehat{G}$ by normal coactions (B, δ) of G and possibly non-continuous Borel-cocycles $\omega \in Z^2(G, \mathbb{T})$!

If G sat. Baum-Connes, $\omega_1 \sim_h \omega_2$, then $K_*(B^{\omega_1}) \cong K_*(B^{\omega_2})$.

Deformation via coactions

Idea for cont. cocycles: Let $\delta : B \to M(B \otimes C^*(G))$ be a normal coaction, and $\omega : G \times G \to \mathbb{T}$ cont. with

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Let $\quad u : G \to UM(B \rtimes_\delta \widehat{G}) : u(g) := \phi(\omega(g, \cdot)). \quad$ Then

$$u_{gh} = \omega(g,h) u_g \alpha_g(u_h) \ \forall g, h \in G \implies \widehat{\delta}_\omega := \mathrm{Ad}\, u \circ \widehat{\delta}$$

is a new (ω-twisted) action on $B \rtimes_\delta \widehat{G}$ s.t. $(B \rtimes_\delta \widehat{G}, \widehat{\delta}_\omega, \phi)$ is still a weak $G \rtimes G$-algebra.

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Define $\quad (B^\omega, \delta^\omega) := ((B \rtimes_\delta \widehat{G})_r^{G,\widehat{\delta}_\omega}, \delta_r^\omega)$. Then

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The case of Borel cocycles is much more technical!

Deformation of μ-coactions

Let (B, δ) be any coaction and let $(B \rtimes_\delta \widehat{G}, \widehat{\delta}, \phi = j_{C_0(G)})$ be the (dual) weak $G \rtimes G$-algebra. Let $\|\cdot\|_\mu$ be the C^*-norm on $C_c(G, B \rtimes_\delta \widehat{G})$ such that (B, δ) is a μ-coaction, i.e.,

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Deformation by Borel cocycles $\omega \in Z^2(G, \mathbb{T})$

Recall: There is a bijection of groups

$$H^2(G, \mathbb{T}) \cong \mathrm{Ext}(G, \mathbb{T}); \quad [\omega] \mapsto [\mathbb{T} \hookrightarrow G_\omega \twoheadrightarrow G]$$

where $\quad G_\omega := G \times \mathbb{T} \quad$ with mult. $\quad (g, z)(h, u) = (gh, \omega(g, h)zu).$

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  This holds for all dual coctions (even for Fell bundles).

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- Via $H^2(G, \mathbb{T}) \cong \mathrm{Br}_G(\mathcal{K}); [\omega] \mapsto [\operatorname{Ad} \rho^\omega]$ we obtain an (equivalent!) deformation via actions $\gamma : G \curvearrowright \mathcal{K}$!

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$$B^\omega \rtimes_{\delta^\omega} \widehat{G} \cong A^\omega \cong A = B \rtimes_\delta \widehat{G} A^\omega = A$$

  This holds for all dual coctions (even for Fell bundles).

- The twisted system $\left(B \rtimes_\delta G, (\widehat{\delta}, \omega)\right)$ always stabilizes to $(B \rtimes_\delta G \otimes \mathcal{K}, \widehat{\beta} \otimes \mathrm{Ad}\,\rho^\omega)$ with $\mathcal{K} := \mathcal{K}(L^2(G))$. Hence

$$\left(B^\omega \rtimes_{\delta^\omega} \widehat{G}, \widehat{\delta^\omega}\right) \sim_M \left(B \rtimes_\delta G \otimes \mathcal{K}, \widehat{\beta} \otimes \mathrm{Ad}\,\rho^\omega\right).$$

- Via $H^2(G, \mathbb{T}) \cong \mathrm{Br}_G(\mathcal{K}); [\omega] \mapsto [\mathrm{Ad}\,\rho^\omega]$ we obtain an (equivalent!) deformation via actions $\gamma : G \curvearrowright \mathcal{K}$!
- Deformation via actions on $\mathcal{K}$ behaves well w.r.t continuous fields of actions $X \ni x \mapsto \gamma_x : G \curvearrowright \mathcal{K}$!

Continuity

By a continuous family of actions $X \ni x \mapsto \gamma_x : G \curvearrowright \mathcal{K}$ we understand a $C_0(X)$-linear action $\gamma : G \curvearrowright C_0(X, \mathcal{K})$, which induces actions $\gamma_x : G \curvearrowright \mathcal{K}$ on the fibres.

Theorem (Buss-E '23) Let $\gamma : G \curvearrowright C_0(X, \mathcal{K})$ be as above and let $\delta : B \to M(B \otimes C^*(G))$ be a μ-coaction for some correspondence cp functor $\rtimes_\mu$.

Then our constructions yield a field of C^*-algebras $\{\mathcal{B}^\gamma : q_x : \mathcal{B}^\gamma_\mu \to B^{\gamma_x}_\mu\}$ together with a $C_0(X)$-linear Morita equivalence

$$\mathcal{B}^\gamma \sim_M ((B \rtimes_\delta \widehat{G}) \otimes C_0(X, \mathcal{K})) \rtimes_{\widehat{\delta} \otimes \gamma, \mu} G.$$

Thus $\mathcal{B}^\gamma$ has the same continuity properties as the crossed product!

Examples

Fell-bundles Suppose that $\mathcal{A} := \dot{\cup}\{A_g : g \in G\}$ is a (continuous) Fell-bundle over G. There is a multiplication and involution

$$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad * : \mathcal{A} \to \mathcal{A}$$

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$$\|f\|_{\max} := \sup\{\|\pi(f)\| : \pi : C_c(\mathcal{A}) \to \mathcal{B}(\mathcal{H}) \text{ is } L^1\text{-cont. } *\text{-rep.}\}$$

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For $\omega \in Z^2(G, \mathbb{T})$ we can 'twist' multiplication on $\mathcal{A}$ by ω to obtain a new Fell-bundle $\mathcal{A}_\omega$. Then $(C^*_\mu(\mathcal{A})^\omega, \delta^\omega) = (C^*_\mu(A_\omega), \delta_\omega)$

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This covers all examples if G is discrete! (Makoto Yamashita).

Examples

Let $G = \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$ then for each $\theta \in \mathbb{R}$ there is a canonical cocycle $\omega_\theta \in Z^2(\mathbb{R},\mathbb{T})$ given by

$$\omega_\theta\left(\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)\right) = e^{\pi\theta(x_1 y_2 - x_2 y_1)}$$

This cocycle is invariant under the action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$, hence ω_θ extends to a 2-cocycle $\widetilde{\omega}_\theta$ on $G = \mathbb{R}^2 \rtimes SL(2,\mathbb{R})$.

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So for any μ-coaction $\delta : B \to M(B \otimes C^*(G))$ (with $\rtimes_\mu$ a correspondence cp functor) we obtain deformations

$$(B_\theta, \delta_\theta) := (B_\mu^{\tilde{\omega}_\theta}, \delta_\mu^{\tilde{\omega}_\theta}), \quad \theta \in \mathbb{R}$$

which are all KK-equivalent.
If we start with a normal coaction, this gives a continuous field of deformed algebras over $\mathbb{R}$!

Examples

Let $G = \mathrm{PSL}(2, \mathbb{R})$. Then $H^2(G, \mathbb{T}) \cong \mathbb{T}$.

So, given a μ-coaction $\delta : B \to M(B \otimes C^*(G))$ for a correspondence c.p. functor $\rtimes_\mu$, we obtain a family of deformed coactions (B_μ^z, δ_μ^z), $z \in \mathbb{T}$.

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Note In case $B = \mathbb{C}$ with trivial coaction, we obtain the twisted group algebras $B^z_\mu = C^*_\mu(G, \omega_z)$.

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Fact: For $z = -1$ one can show that $C^*_{\max}(G, \omega_{-1}) = C^*_r(G, \omega_{-1})$. This follows from the fact that the representations of these algebras correspond to the unitary reps of $\mathrm{SL}(2,\mathbb{R})$ which restrict to the non-trivial character of the center $\mathbb{Z}_2 = Z(\mathrm{SL}(2,\mathbb{R}))$.

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Note In case $B = \mathbb{C}$ with trivial coaction, we obtain the twisted group algebras $B_\mu^z = C_\mu^*(G, \omega_z)$.

Fact: For $z = -1$ one can show that $C_{\max}^*(G, \omega_{-1}) = C_r^*(G, \omega_{-1})$. This follows from the fact that the representations of these algebras correspond to the unitary reps of $\mathrm{SL}(2, \mathbb{R})$ which restrict to the non-trivial character of the center $\mathbb{Z}_2 = Z(\mathrm{SL}(2, \mathbb{R}))$.

Thus the deformation via $\rtimes_{\max}$ deforms $C_{\max}^*(G)$ to $C_{\max}^*(G, \omega_{-1})$ and deformations via $\rtimes_r$ deforms $C_r^*(G)$ to $C_r^*(G, \omega_{-1}) = C_{\max}^*(G, \omega_{-1})$.

Some References

[1] J. Bhowmick, S. Neshveyev, and A. Sangha, *Deformation of operator algebras by Borel cocycles*, J. Functional Analysis **265** (2013), 983–1001.

[2] A. Buss and S. Echterhoff, *Universal and exotic generalized fixed-point algebras for weakly proper actions and duality*, Indiana Mathematics J. **63** (2014), no. 6, 1659–1701.

[3] ______, *Rieffel proper actions*, Journal of Operator Theory **75** (2016), no. 1, 49–73.

[4] ______, *A new approach to (exotic) deformation of C*-algebras via coactions*. arXiv:2305.09997.

[5] A. Buss, S. Echterhoff, and S. Willett, *Exotic crossed products and the Baum-Connes conjecture*, J. Reine Angew. Math. **740** (2018), 111–159.

[6] Pavel Kasprzak, *Rieffel deformation via crossed products*, J. Funct. Anal. **257** (2009), no. 5, 1288–1332.

[7] S. Neshveyev and L. Tuset, *Deformation of* C^**-algebras by cocycles on locally compact quantum groups*, Adv. Math. **254** (2014), 454–496.